LECTURE 13: SPINORS

The Dirac operators we have defined so far all depend on the choice of a Clifford bundle equipped with a compatible connection. We shall now try to define a Dirac operator which does not depend so much on such choices. For this we first need a special representation of the Clifford algebra.

1. Spinors and double coverings of the orthogonal groups

1.1. **The spin representation of the Clifford algebra.** Fermions in physics are not just elements of any representation of the Clifford algebra, they are elements of the *spin representation*:

Proposition 1.1. *Suppose that $n := p + q$ is even. Then* $\mathrm{Cliff}_{p,q}$ *has a unique irreducible representation on a vector space $\mathbb{S}$ of dimension $2^{n/2}$, called the* spin representation*.*

For students who have followed a course on representation theory of finite groups, we sketch the argument proving this proposition: In $\mathrm{Cliff}_{p,q}$ the basis elements (**??**) and their negatives

$$G := \{\pm 1, \pm\psi_{i_1} \cdots \psi_{i_k},\ 1 \leq i_1 < \ldots < i_k \leq n,\ k = 1, \ldots, n\}.$$

define a group. This follows easily from the Clifford relations (**??**), and we see that G is a group with 2^{n+1} elements. It also follows from the same Clifford relations that -1 is central in G and that $G/\{-1\}$ is an abelian group. This abelian group has 2^n elements, and therefore 2^n inequivalent irreducible representations. The representations also define representations of G (through the quotient homomorphism $G \to G/\{-1\}$), so the question is: how much more does G have? For this, we analyse the conjugacy classes in G: there are $2^n - 1$ conjugacy classes of the form $\{\gamma, -\gamma\}$ with γ an element corresponding to $k > 0$ as above, and the two central elements $\{1\}$ and $\{-1\}$ each form a conjugacy class, so we have a total of $2^n + 1$ conjugacy classes. This means that G has one extra representation, and this is the one we are interested in!

Anticipating the result, let's call this representation M. By the usual order-degree relation for finite groups, its dimension is determined by

$$2^n + \dim(M)^2 = 2^{n+1},$$

so $\dim(M) = 2^{n/2}$. Now, since this representation does not come from one of the representations of $G \to G/\{-1\}$, the element $-1 \in G$ will act as multiplication by -1 (not $+1$), and it is not difficult to show that on this representation space, the action of G extends to an action of $\mathrm{Cliff}_{p,q}$.

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1.2. **The group** Spin(p,q)**.** We have seen in §**??** that Dirac's description of spin 1/2 particles inevitably resulted in a representation of a double covering of the Lorentz group, rather than the Lorentz group itself. This is in fact a generic feature associated to Clifford algebras, as we shall now see. As usual, we write $n = p + q$, and we consider the group of invertible isometries of $\mathbb{R}^{p,q}$:

$$O(p,q) := \{g \in GL(n,\mathbb{R}),\ \langle gx, gy\rangle = \langle x,y\rangle\ ,\ \text{for all } x,y \in \mathbb{R}^{p,q}\}$$

This group contains a subgroup of transformations with determinant 1:

$$SO(p,q) = \{g \in O(p,q),\ \det(g) = 1\}.$$

These are examples of *Lie groups* meaning that they have both a group structure as well as a smooth manifold structure and the two are compatible. We have already mentioned that $SO(3,1)$ is the Lorentz group.

From the point of view of Clifford algebras, these groups appear naturally as their symmetry groups. Indeed, let $g \in O(p,q)$, then the assignment

$$v \mapsto \psi^g(v) := \psi(g^{-1}v), \quad v \in \mathbb{R}^{p,q}$$

satisfies the Clifford relations (**??**) as well:

$$\psi^g(v)\psi^g(w) + \psi^g(w)\psi^g(v) = -2\eta(g^{-1}v, g^{-1}w) = -2\eta(v,w).$$

In terms of the basis $\{e_i\}$ of $\mathbb{R}^{p,q}$, g is given by the matrix (g_{ij}) with inverse (g^{ij}) and this leads to the transformation[1]

$$\psi_i \mapsto \sum_j g^{ij}\psi_j. \tag{1}$$

Inside the Clifford algebra $\mathrm{Cliff}_{p,q}$ we now define

$$\begin{aligned}\mathrm{Pin}(p,q) &:= \{\psi(x_1)\cdots\psi(x_k),\ ||x_i||^2 = \pm 1,\ \text{for all } i = 1,\ldots,k\},\\ \mathrm{Spin}(p,q) &:= \{\psi(x_1)\cdots\psi(x_k) \in \mathrm{Pin}(p,q),\ k \text{ is even}\}.\end{aligned}$$

Lemma 1.2. *Clifford multiplication induces a group structure on* Pin(p,q) *and* Spin(p,q) *turning them into Lie groups.*

Proof. Concerning the group structure, the only thing that is not immediately clear is the existence of inverse. But remark that $\psi(x)$, $||x|| = \pm 1$ is equal to plus or minus its own inverse, so each element in Pin(p,q) is a product of invertible elements, and hence invertible itself. We will skip the statement about the Lie structure. □

Proposition 1.3. *There are short exact sequence of groups*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(p,q) \xrightarrow{\rho} O(p,q) \longrightarrow 1,$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(p,q) \xrightarrow{\rho} SO(p,q) \longrightarrow 1.$$

[1]Mathematically, what we are describing here is a group homomorphism $O(p,q) \to \mathrm{Aut}(\mathrm{Cliff}_{p,q})$.

Remark 1.4. For $p = 3$, $q = 1$, one can show that $\mathrm{Spin}(3,1) \cong SL(2,\mathbb{C})$, so we recover the double covering (**??**) of the Lorentz group. Also, for $p \geq 3$, $\pi_1(SO(p)) = \mathbb{Z}_2$, and therefore in that case $\mathrm{Spin}(p)$ is the *universal covering*. For $p = 3$, we have $\mathrm{Spin}(3) \cong SU(2)$.

Proof. Let us first describe the homomorphism $\rho : \mathrm{Pin}(p,q) \longrightarrow O(p,q)$. Denote by $\epsilon : \mathrm{Cliff}_{p,q} \to \mathrm{Cliff}_{p,q}$ the map which sends

$$\epsilon(\psi_1 \cdots \psi_k) = \begin{cases} \psi_1 \cdots \psi_k & k \text{ even}, \\ -\psi_1 \cdots \psi_k & k \text{ odd}. \end{cases}$$

(This map is well defined because the defining relations in the Clifford algebra (**??**) are purely even.) For $\tilde{g} \in \mathrm{Pin}(p,q)$, there is a unique $g \in O(p,q)$ such that

$$\tilde{g}\psi(v)\epsilon(\tilde{g}^{-1}) = \psi(g^{-1}v), \quad \text{for all } v \in \mathbb{R}^{p,q}.$$

To see this, take $\tilde{g} = \psi(x)$ with $||x|| = \pm 1$, with inverse $\tilde{g}^{-1} = -||x||^2\psi(x)$. Then

$$\begin{aligned} \epsilon(\tilde{g})\psi(v)\tilde{g}^{-1} &= ||x||^2\psi(x)\psi(v)\psi(x) \\ &= ||x||^2(-\psi(x)^2\psi(v) - 2\psi(x)\eta(x,v)) \\ &= \psi(v) - 2\frac{\eta(x,v)}{\eta(x,x)}\psi(x) \\ &= \psi\left(v - 2\frac{\eta(x,v)}{\eta(x,x)}x\right). \end{aligned}$$

Now the map $v \mapsto v - 2\frac{\eta(x,v)}{\eta(x,x)}x$ is just the reflection in the hyperplane defined by $\eta(x,v) = 0$. It is a Theorem of Cartan–Dieudonné that such reflections generate the group $O(p,q)$. This defines the homomorphism ρ. Any element of the subgroup $SO(p,q)$ is given by a composition of an even number of reflections.

Suppose that $\tilde{g} \in \ker\rho$. Then

$$\tilde{g}v - \epsilon(\tilde{g})v\tilde{g} = v, \quad \text{for all } v \in \mathbb{R}^{p,q}.$$

Since $\mathbb{R}^{p,q}$ generates the whole Clifford algebra $\mathrm{Cliff}_{p,q}$, this means that $\tilde{g} \in \mathrm{Pin}(p,q) \cap \mathbb{R} = \mathbb{Z}_2$. This shows exactness of the sequence.

To show that the covering is *non-trivial*, show that in $\mathrm{Spin}(p,q)$ there is a path connecting -1 and 1: choose two orthonormal vectors $e_1, e_2 \in \mathbb{R}^{p,q}$ with $||e_1||^2 = ||e_2||^2 = \pm 1$. With this, the path

$$\begin{aligned} \gamma(s) &= \pm\cos(2s) + \psi(e_1)\psi(e_2)\sin(2s) \\ &= (\psi(e_1)\cos(s) + \psi(e_2)\sin(s))(\psi(e_1)\cos(s) - \psi(e_2)\sin(s)), \end{aligned}$$

satisfies $\gamma(0) = \pm 1$ and $\gamma(\pi) = \mp 1$. $\square$

2. Projective representations and Quantum Mechanics

You may wonder how bad it actually is that with the theory of Clifford algebras we only find a representation of a double covering of the Lorentz (and Poincaré) group, after all Lorentz-invariance is main point of the theory of relativity! So, here are some reassuring words.

In Quantum Mechanics, the state of a system is described by a non-zero vector $\psi \in \mathcal{H}$ in a Hilbert space of unit norm $||\psi|| = 1$, but it is well-known that two vectors that differ by a phase describe the same state. Apparently the space of pure states is given by the quotient

$$\mathbb{P}(\mathcal{H}) := \{\psi \in \mathcal{H},\ ||\psi|| = 1\}/\mathbb{T},$$

also known as the *projective Hilbert space*. The structure of this space relevant for physics is the map $p : \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0,1]$ known as the *transition probability*:

$$p(\psi_1, \psi_2) := |\langle \psi_1, \psi_2 \rangle|^2.$$

A unitary operator $U : \mathcal{H} \to \mathcal{H}$ induces a map $\mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ preserving p. Conversely, a famous theorem of Wigner says that any invertible map from $\mathbb{P}(\mathcal{H})$ to $\mathbb{P}(\mathcal{H})$ preserving p must come from a unitary or anti-unitary linear operator on $\mathcal{H}$.

The map induced on $\mathbb{P}(\mathcal{H})$ by $U \in U(\mathcal{H})$ is not unique: again two maps that differ by a phase induce the same transformation. This leads to the *projective unitary group*

$$PU(\mathcal{H}) := U(\mathcal{H})/\mathbb{T},$$

so that we have an exact sequence

$$1 \to \mathbb{T} \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1.$$

A *unitary representation* of a group G is simply a homomorphism $G \to U(\mathcal{H})$. Likewise, a *projective unitary representation* is a homomorphism $G \to PU(\mathcal{H})$, and the preceding discussion makes clear that there is nothing wrong with those from the point of view of physics, even more they are completely natural!

We now see that if we consider the spinor representation $\mathbb{S}$ of $\mathrm{Cliff}_{p,q}$, we get, by restriction, a representation of the group $\mathrm{Spin}(p,q)$, and one checks that $\mathbb{Z}_2$ acts by ± 1. It therefore induces a projective representation of $SO(p,q)$, and we have maps

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathrm{Spin}(p,q) \arrow[r] \arrow[d] & SO(p,q) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r] & U(\mathbb{S}) \arrow[r] & PU(\mathbb{S}) \arrow[r] & 1
\end{tikzcd}$$

3. THE SPIN-DIRAC OPERATOR

3.1. **Spin structures.** Consider an even dimensional, oriented, pseudo-riemannian manifold (M, g) of signature (p, q). Let us try to construct the spinor bundle as follows: consider an atlas of local charts $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$. We assume the underlying covering $\{U_\alpha\}_{\alpha\in I}$ is *good*, meaning that all intersections are contractible. The local charts define local trivializations so that we can construct the tangent bundle as in Remark **??**, using a system of transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to SO(p, q)$. This in turn defines local trivializations of the Clifford bundle $\mathrm{Cliff}(TM)|_{U_\alpha} \cong U_\alpha \times \mathrm{Cliff}_{p,q}$, with transition functions given by the same $SO(p, q)$-valued functions $\varphi_{\alpha\beta}$ acting via (1).

To construct the spinor bundle $\mathscr{S}$ we use the local pieces $U_\alpha \times \mathbb{S}$ and try to find a "cocycle" $\tilde{\varphi}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}(p, q)$ satisfying the conditions (**??**). So let us choose a covering $\tilde{\varphi}_{\alpha\beta}$ of the $SO(p, q)$-valued cocycle $\varphi_{\alpha\beta}$ under the covering map of Proposition 1.3, making sure we satisfy the first two conditions of (**??**). To examine the third condition, remark that on triple overlaps we have

$$\rho(\tilde{\varphi}_{\alpha\beta}\tilde{\varphi}_{\gamma\alpha}\tilde{\varphi}_{\beta\gamma}) = 1,$$

and therefore

$$\tau_{\alpha\beta\gamma} := \tilde{\varphi}_{\alpha\beta}\tilde{\varphi}_{\gamma\alpha}\tilde{\varphi}_{\beta\gamma} = \pm 1.$$

These "sign elements" $\{\tau_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma\in I}$ define a cohomology class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ described[2]by the following singular 2-cocycle: for any singular 2-simplex $\sigma : \Delta^2 \to M$, choose α, β, γ such that its vertices (ordered) lie in U_α, U_β and U_γ. To such a simplex we assign the element $\tau_{\alpha\beta\gamma}$. One can check that it is independent of the chosen $\alpha, \beta, \gamma \in I$ and extending linearly, we obtain a singular 2-cochain $\tilde{\tau} \in S^2(M, \mathbb{Z}_2)$. Clearly, we have

$$\delta\tilde{\tau} = 0,$$

since in the boundary of a 3-simplex each line segment appears twice, and line segments correspond to choices of lifts over intersections, combined with the fact that in $\mathbb{Z}_2$ any element squares to 1. The resulting cohomology class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ is called the *second Stiefel–Whitney class*. It is an obstruction to the existence of spin structures:

Theorem 3.1 (Haefliger)**.** *A spin structure on M exists if and only if the second Stiefel–Whitney class vanishes: $w_2(M) = 0$. In that case, different spin structures are classified by the cohomology group $H^1(M, \mathbb{Z}_2)$.*

We should emphasize that vanishing of the second Stiefel–Whitney classes is a very mild condition, in fact in many cases one just shows that $H^2(M, \mathbb{Z}_2) = 0$ altogether. However, even when the second Stiefel–Whitney class vanishes, one has to be aware that a spin structure amounts to a *choice* of a lift, and those are parameterized (up to isomorphism) by $H^1(M, \mathbb{Z}_2)$.

[2] If you have seen Čech cohomology before you may verify at once that $\{\tau_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma\in I}$ is a Čech-cocycle.

In any case, the choice of a spin structure amounts to the construction of the spinor bundle $\mathscr{S} \to M$, with fiber $\mathscr{S}_x \cong \mathbb{S}(T_x M)$ and comes equipped with a canonical connection compatible with the Clifford action. These are enough ingredients for the construction of the spin-Dirac operator D acting on sections of $\mathscr{S}$.

As we have seen, the Dirac operator is elliptic in the riemannian case, so we can apply the Atiyah–Singer index theorem, resulting in:

$$\text{index}(D^+) = \int_M \hat{A}(M),$$

where $\hat{A}(M)$ is the so-called "A-hat genus", a characteristic class determined by the power series expansion

$$\det\left(\frac{A/2}{\sinh(A/2)}\right).$$

By the splitting principle, this is given by

$$\hat{A}(M) = \prod_i \frac{x_i/2}{\sinh(x_i/2)}.$$

Underlying this definition is the fact that the function

$$f(z) := \frac{z/2}{\sinh(z/2)},$$

is evidently analytic in a neighborhood of the singular point $z = 0$, and satisfies

$$\lim_{z\to 0} f(z) = 1.$$

By the removable singularity theorem, the function is therefore analytic in $z = 0$ and has a power series expansion

$$\frac{z/2}{\sinh(z/2)} = 1 - \frac{1}{24}z^2 + \mathcal{O}(z^4). \tag{2}$$

3.2. **Coupling to vector bundles.** Let M be an even dimensional spin manifold and $E \to M$ a vector bundle. It is possible to "couple" the Dirac operator to this vector bundle, constructing a first order differential operator D_E acting on sections of $\mathscr{S} \otimes E$. This is not difficult at all: the tensor product $\mathscr{S} \otimes E$ carries a representation of $\text{Cliff}(TM)$ as well, acting trivially on the second components as $\psi(X) \otimes 1_E$. Choosing any connection ∇^E on E defines a connection $\nabla^{\mathscr{S}} \otimes 1 + 1 \otimes \nabla^E$ compatible with the Clifford action, and then our usual construction in Definition **??** yields the Dirac operator D_E. In this case the Atiyah–Singer index theorem gives

$$\text{index}(D_E^+) = \int_M \hat{A}(M)\text{ch}(E). \tag{3}$$

Remark 3.2. The Dirac operator D_E involves the choice of a connection ∇^E, but one can easily check that its principal symbol $\sigma_1(D_E^+)$ does *not* depend on ∇^E. Therefore the index is independent of this choice, and this is confirmed on the right hand side of the index theorem by the independence of the Chern character of the connection.

From the point of view of physics, this construction can be viewed as "coupling fermions to gauge fields": if we have a gauge theory with gauge fields defined as connections A on a principal G-bundle $P \to M$, then the construction above allows us to define a Dirac operator D_A acting on sections of $\mathscr{S} \otimes E(V)$, where $E(V)$ is the vector bundle associated to a representation of G.

3.3. **Examples.**

Example 3.3. Consider the 2-torus $\mathbb{T}^2$ obtained by considering two flat periodic coordinates x, y defined mod $\mathbb{Z}$. It is not difficult to see that for the torus the second Stiefel–Whitney class is zero (as for any two-dimensional oriented surface), so a spin structure exists. Recall that $\mathrm{Cliff}_{2,0} \cong \mathbb{H}$ and one can show that the spinor representation is given by $\mathbb{C}^2$, with $\mathbf{i}$, j and k acting by i times the Pauli spin matrices σ_1, σ_2 and σ_3. Because the torus is flat, we do not have to choose connections, but can just use the exterior derivative. This leads to the spin-Dirac operator

$$D = i \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix}.$$

We recognize the off-diagonal entries as the Cauchy–Riemann operator ∂ and $\bar{\partial}$ in the complex coordinate $z = x + iy$. Since $z \mapsto f(z)$ above is an even function, the power series expansion is an expansion in z^2, so $\hat{A}(M) = 1$ for a two-dimensional manifold. We can couple the spin-Dirac operator above to a vector bundle with connection (E, ∇^E) and then the index theorem (3) gives

$$\mathrm{index}(D_E^+) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} \mathrm{Tr}(F(\nabla^E)).$$

This example generalizes almost verbatim to any oriented two-dimensional surface. This is not quite the Cauchy–Riemann operator on a Riemann surface that we discuss below.

Example 3.4 (The Cauchy–Riemann operator)**.** The following is *not* an example of a spin-Dirac operator, but rather of an elliptic complex: Let Σ be a compact Riemann surface. Topologically, we know that $H^2_{dR}(\Sigma) = \mathbb{R}$ with generator σ dual to the fundamental class $[\Sigma] \in H_2(\Sigma)$. The complex structure on Σ induces a splitting of the complexified tangent bundle $T\Sigma \otimes \mathbb{C} = T^{(1,0)}\Sigma \oplus T^{(0,1)}\Sigma$ and its dual, the cotangent bundle. We denote by $\Omega^{0,1}(\Sigma)$ the differential 1-forms that can locally be written as $\alpha = f(z, \bar{z})d\bar{z}$ in a local holomorphic coordinate z. We then consider the map

$$C^\infty(\Sigma) \xrightarrow{\bar{\partial}} \Omega^{(0,1)}(\Sigma).$$

It is not difficult to show that this operator is elliptic, so that the index theorem gives

$$\mathrm{index}(\bar{\partial}) = -\int_\Sigma \frac{\mathrm{ch}(\underline{\mathbb{C}} - (T^{(0,1)}\Sigma)^*)\mathrm{td}(T\Sigma \otimes \mathbb{C})}{e(T\Sigma)}.$$

Let us work out the left and right hand side. On the left we see that $\ker(\bar{\partial})$ are precisely the holomorphic functions $\mathcal{O}(\Sigma)$, because $\bar{\partial} = \frac{\partial}{\partial \bar{z}} \otimes d\bar{z}$. Similarly, for the cokernel, Dolbeault's theorem gives $\mathrm{coker}(\bar{\partial}) \cong \Omega^1_{\mathrm{hol}}(\Sigma)$, the space of holomorphic 1-forms, i.e., those that are locally of the form $f(z)d\bar{z}$, with $f(z)$ holomorphic.

On the right hand side, write $x = c_1(T^{(1,0)}\Sigma)$. Then we have

$$\begin{aligned} e(T\Sigma) &= c_1(T^{(0,1)}\Sigma) = x \\ \mathrm{td}(T\Sigma \otimes \mathbb{C}) &= \mathrm{td}(T^{(1,0)}\Sigma)\mathrm{td}(T^{(0,1)}\Sigma) = \frac{x}{1-e^{-x}}\frac{-x}{1-e^{x}} \\ \mathrm{ch}(\underline{\mathbb{C}} - (T^{(0,1)}\Sigma)^*) &= 1 - e^{-x} \end{aligned}$$

These equations therefore combine to

$$\frac{\mathrm{ch}(\underline{\mathbb{C}} - (T^{(0,1)}\Sigma)^*)\mathrm{td}(T\Sigma \otimes \mathbb{C})}{e(T\Sigma)} = \frac{x}{1-e^{x}} = 1 + \frac{1}{2}x$$

It is a fundamental fact that on a Riemann surface $x = (2-2g)\sigma$, where g is the genus of Σ. Combining everything, we now find

$$\dim(\mathcal{O}(\Sigma)) - \dim(\Omega^1_{\mathrm{hol}}(\Sigma)) = 1 - g,$$

a special case of the famous Riemann–Roch theorem. To get the full Riemann–Roch formula, we can add a divisor D to the picture: there is an associated holomorphic line bundle L_D with $c_1(L_D) = \deg(D)\sigma$. The left hand side now counts meromorphic functions and 1-forms with poles controlled by the divisor D, and the Atiyah–Singer index theorem gives

$$\dim(\mathcal{O}(\Sigma)) - \dim(\Omega^1_{\mathrm{hol}}(\Sigma)) = \deg(D) + 1 - g.$$

Example 3.5. Consider the 2-sphere S^2. As an orientable surface, it has a spin structure and therefore an associated spin-Dirac operator D. Consider now the principal $U(1)$-bundle P with connection (gauge field A) over S^2 of the Dirac monopole. The spin-Dirac operator D_A coupled to this gauge field has index given by

$$\mathrm{index}(D_A^+) = \frac{1}{2\pi}\int_{S^2} F(A).$$

As we have seen, the right hand side is equal to the winding number in $\pi_1(U(1)) = \mathbb{Z}$ that defines the principle bundle P.

Example 3.6. Now we consider the four-sphere S^4. This manifold is also spin, and one can show that it has Dirac genus $\hat{A}(S^4) = 1$. Consider now the principal $SU(2)$-bundle P over S^4 associated to the instanton solution of the Yang–Mills equations. This time the index theorem for the spin Dirac operator D_A coupled to this gauge field gives

$$\mathrm{index}(D_A^+) = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}(F(A) \wedge F(A)).$$

This time the integer on the right hand side equals the class in $\pi_3(SU(2)) = \mathbb{Z}$ that determines the isomorphism class of P: we can define P by gluing two trivial $SU(2)$-bundles over the northern and southern hemispheres U and V: These are glued together by a transition function (i.e., a *local gauge transformation* in the language of physicists) $\varphi : U \cap V \to SU(2)$. Since $U \cap V \sim_h S^3$ this determines a unique element in $[S^3, SU(2)] = \pi_3(SU(2))$.